Reconstructing Refractive Index Discontinuities from Truncated Phase-Contrast Projections

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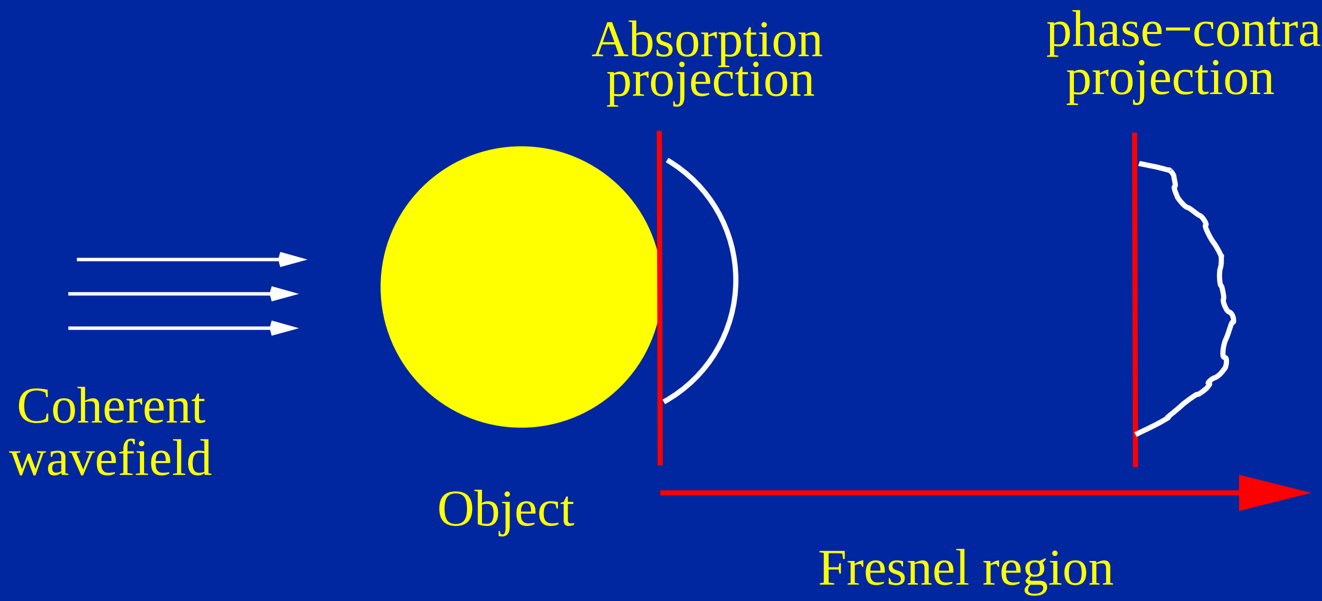
Outline

- Brief introduction to phase-contrast tomography
- A backprojection-only local tomography algorithm
- Interpretation of phase-contrast local tomography algorithms
- Reconstructing the magnitude of refractive index jumps.

Background: Phase-Contrast Tomography

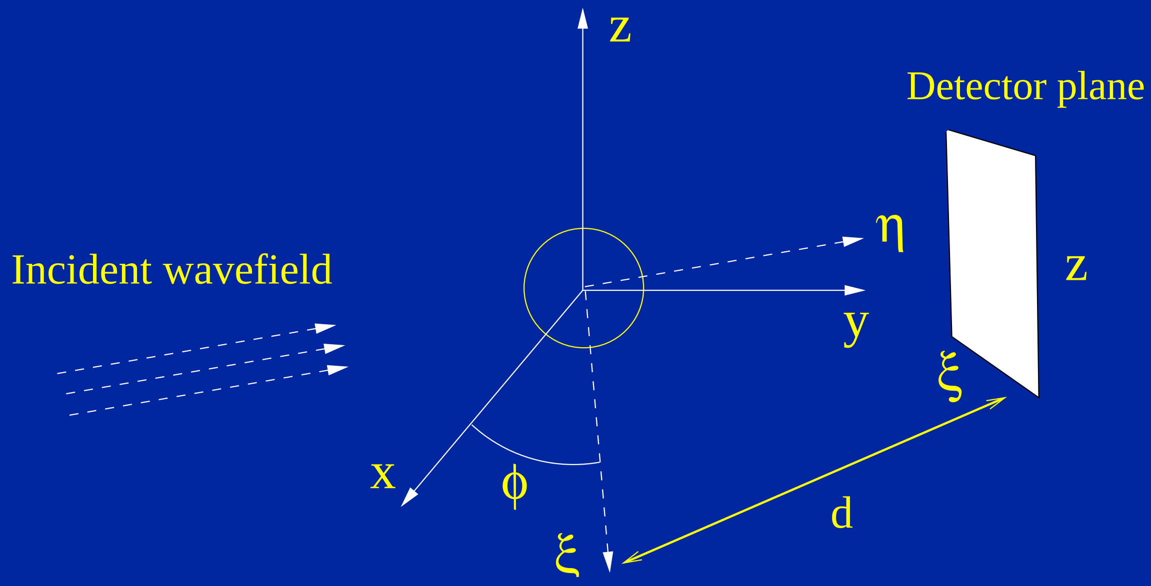
- When imaging with a spatially coherent X-ray beam, phase-contrast projection images are obtained:



- Phase-contrast tomography seeks to reconstruct the refractive index from a set of phase-contrast projections.

Phase-Contrast Tomography: Geometry

- Let the z-axis of a reference system (x, y, z) define the rotation axis for tomographic scanning.
- The rotated coordinate system (ξ, η, z) is related to the reference system by $\xi = x\cos\phi + y\sin\phi$, $\eta = y\cos\phi - x\sin\phi$



Phase-Contrast Tomography: Measurement Data

- Let $I^{\eta=d}(\xi, z, \phi)$ denote measured projections.
- It can be shown that under near-field conditions [Bronnikov]:

$$I^{\eta=d}(\xi, z, \phi) = I^{\eta=0}(\xi, z, \phi)\left[1 - \frac{\lambda d}{2\pi}\nabla^2\Phi(\xi, z, \phi)\right],$$

where

$$\Phi(\xi, z, \phi) = \frac{2\pi}{\lambda}\int_{\mathbb{R}^2} dxdy\; a(x, y, z)\; \delta(\xi - x\cos\phi - y\sin\phi),$$

and $a(x, y, z) = n(x, y, z) - 1$ is the object function.

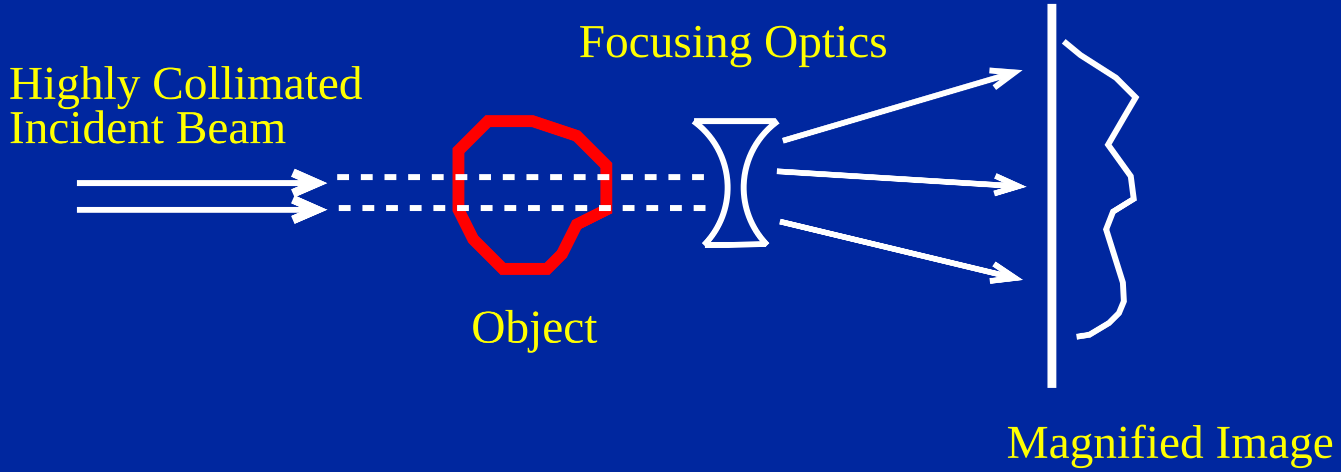
The Global Reconstruction Problem

- Goal: Reconstruct $a(x, y, z)$ from knowledge of $I^{\eta=d}(\xi, z, \phi)$.
- Bronnikov has proposed an exact analytic reconstruction algorithm.
- In practice, the FBP algorithm is typically used to reconstruct an edge-map image.
- Both of these approaches assume untruncated projection data.

Phase-Contrast Local Tomography

- In many applications, the measured phase-contrast projections are truncated.

- In synchrotron imaging, the FOV may be intentionally reduced to increase spatial resolution.



Phase-Contrast Local Tomography

Questions:

- Can $\mathcal{F}a(x, y, z)$ be exactly reconstructed from truncated phase-contrast projections?
- If yes, what is the operator $\mathcal{F}$?
- Simple reconstruction algorithms?
- To answer these questions, we can utilize conventional local tomography reconstruction theory.

Local Tomography Theory

- Local tomography methods have been developed for absorption CT.
- Local tomography seeks to reconstruct, not $\mu(x, y, z)$, but a filtered image denoted by $\Lambda\mu(x, y, z)$
- The operator $\Lambda = \sqrt{-\nabla^2}$ amplifies discontinuities in the reconstructed image.
- $\Lambda\mu(x, y, z)$ can be reconstructed from truncated (i.e., local) projection data.

(1) A Backprojection-Only Local Algorithm for Phase-Contrast Tomography

- Consider a *data function* $m(\xi, z, \phi)$:

$$m(\xi, z, \phi) = \frac{1}{d}\left[\frac{I^{\eta=d}(\xi, z, \phi)}{I^{\eta=0}(\xi, z, \phi)} - 1\right],$$

- By the definition of our imaging transform one obtains immediately:

$$m(\xi, z, \phi) = -\frac{\lambda}{2\pi}\nabla^2_{\xi,z}\Phi(\xi, z, \phi) = -\nabla^2_{\xi,z}\, R\, a(x, y, z),$$

where the 2D Radon transform $R\, a(x, y, z) \equiv p(\xi, \phi; z)$ acts on 2D planes that are perpendicular to the z-axis.

A BP-only Local Tomography Algorithm

- Consider $b(x, y, z)$ that is formed by simply backprojecting $m(\xi, z, \phi)$:

$$b(x, y, z) = \int_0^\pi d\phi \; m(\xi, z, \phi)\big|_{\xi = x\cos\phi - y\sin\phi}.$$

Noting that $-\nabla^2 = \Lambda^2_{\xi,z}$ one obtains

$$b(x, y, z) = \int_0^\pi d\phi \; \Lambda^2_{\xi,z} p(\xi, \phi; z)\big|_{\xi = x\cos\phi - y\sin\phi}.$$

- This formula looks similar to the 2D Lambda FBP algorithm.

A BP-only Local Tomography Algorithm

- It can be shown that

$$b(x,y,z) = \Lambda^2_{x,y,z} a(x,y,z) ** \frac{1}{\sqrt{x^2+y^2}}$$

- A simple backprojection of $m(\xi, z, \phi)$ yields a filtered version of $\Lambda^2_{x,y,z} a(x,y,z)$.
- The $\frac{1}{\sqrt{x^2+y^2}}$ blurring is due to the 2D backprojection operation.

A BP-only Local Tomography Algorithm

Observations:

- The reconstruction is local - no explicit filtering involved.
- The wave propagation physics performs the necessary projection filtering.
- For 2D objects, $\Lambda a(x, y, z)$ can be reconstructed exactly.
- When projections are acquired over a 4π solid angular range, $\Lambda a(x, y, z)$ can be reconstructed exactly.

(2) Direct Application of Λ Tomography

- Another reconstruction strategy is to apply a Λ tomography reconstruction algorithm directly to $m(\xi, z, \phi)$.

- By definition, this reconstruction approach is a local one.

- Here we interpret the reconstructed image mathematically to reveal its precise meaning.

Direct Application of Λ Tomography

- Consider the application of the 2D Λ FBP algorithm:

$$a_{lfbp}(x,y,z) = -\frac{1}{(2\pi)^4}\int_0^{\pi} d\phi\, \frac{\partial^2}{\partial \xi^2} m(\xi,z,\phi)\,|_{\xi = x\cos\phi - y\sin\phi}\ ,$$

  Let $A_{lfbp}(\vec{\nu}) \equiv \mathcal{F}_3\{a_{lfbp}(x,y,z)\}$

- It can be shown that

$$A_{lfbp}(\vec{\nu}) = \mathcal{F}_3\{\Lambda^2 a(x,y,z)\}\,\sqrt{\nu_x^2+\nu_y^2}$$

Direct Application of Λ Tomography

- As with the BP-only algorithm, the 2D Λ FBP algorithm reconstructs an approximation of $\Lambda^2 a(x, y, z)$.

- The high-frequency components of $A_{lfbp}(\vec{\nu})$ in the ν_x-ν_y plane will be amplified by the term $\sqrt{\nu_x^2 + \nu_y^2}$.

- This will result in significantly amplified high-frequency noise components.

- For 2D objects, this reconstruction approach yields $a_{lfbp}(x, y, z) = \Lambda^3 a(x, y, z)$ exactly.

(3) Direct Application of the FBP Algorithm

- In many practical applications, the 2D FBP algorithm is applied directly to the phase-contrast data $m(\xi, z, \phi)$.

- When the projections are untruncated, one reconstructs $\Lambda^2 a(x, y, z)$ exactly (Cloetens, 1997).

- Empirical findings have suggested that, for many object functions, the FBP algorithm may be capable of reconstructing an image from truncated phase-contrast projections that closely approximates $\Lambda^2 a(x, y, z)$.

  Why?

Direct Application of the FBP Algorithm

- For simplicity, we consider here the 2D problem.

- Let the projections be truncated to detector coordinates $|\xi| \leq \xi_r$

  I.e., $m(\xi, \phi) = 0$ for $|\xi| \geq \xi_r$.

- Reconstructed image:

$$a_r(x, y) = \frac{1}{4\pi^2} \int_0^{\pi} d\phi \int_{-\xi_r}^{\xi_r} d\xi' \frac{1}{\xi - \xi'} \left[\frac{\partial m(\xi, \phi)}{\partial \xi} \right]_{\xi = \xi'}$$

Direct Application of the FBP Algorithm

- We can decompose $a_r(x, y)$ into two terms:

$$a_r(x, y) = \underbrace{\frac{1}{4\pi^2} \int_0^\pi d\phi \int_{-\infty}^{\infty} d\xi' \, h_{\xi_r}(\xi - \xi') \left[\frac{\partial m(\xi, \phi)}{\partial \xi} \right]_{\xi = \xi'}}_{a_{pl}(x,y)}$$

$$\underbrace{- \frac{1}{4\pi^2} \int_0^\pi d\phi \left\{ \int_{\xi - 2\xi_r}^{-\xi_r} d\xi' + \int_{\xi_r}^{\xi + 2\xi_r} d\xi' \right\} \frac{1}{\xi - \xi'} \left[\frac{\partial m(\xi, \phi)}{\partial \xi} \right]_{\xi = \xi'}}_{e(x,y)},$$

where $h_{\xi_r}(\xi) \equiv \frac{1}{\xi}$ for $|\xi| \leq 2\xi_r$ and $h_{\xi_r}(\xi) \equiv 0$ otherwise.

Direct Application of the FBP Algorithm

- Consider the term $a_{pl}(x,y)$:

  It can be shown that

$$a_{pl}(x,y) = E_{\xi_r}(r) ** \Lambda a(x,y),$$

  where $E_{\xi_r}(r) \equiv R^{-1}\frac{\partial m_{\xi_r}(\xi)}{\partial \xi}$.

  When $\xi_r \to \infty$, $E_{\xi_r}(r) \to \mathcal{F}_2^{-1}\{|\vec{\nu}|\}$

  Therefore $a_{pl}(x,y) = E_{\xi_r}(r) ** \Lambda a(x,y) \to \Lambda^2 a(x,y)$

Direct Application of the FBP Algorithm

- Consider the term $e(x, y)$.

- $e(x, y)$ describes how structures outside the ROI can influence the reconstruction of $a_{pl}(x, y)$ inside the ROI.

- When $\xi_r \to \infty$, $e(x, y) \to 0$, $a_r(x, y) \to \Lambda^2 a(x, y)$.

- It can be shown that $e(x, y)$ is generally small except when there are a large number of discontinuities in $a(x, y)$ that reside outside, but close to, the imaged ROI.

Direct Application of the FBP Algorithm

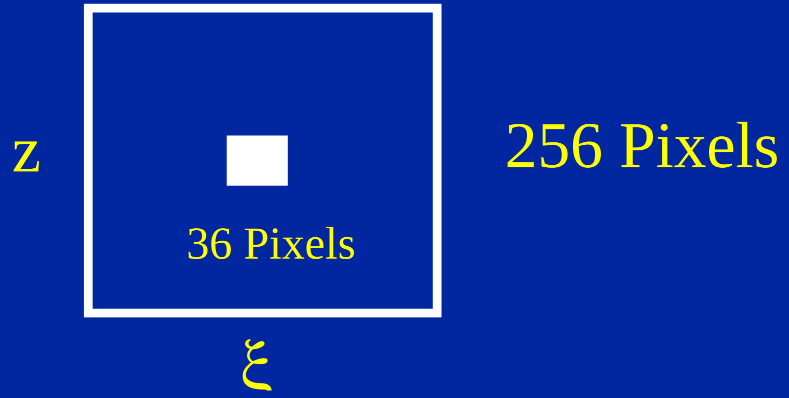
- For many objects of practical interest, the FBP algorithm may behave ‘almost locally’.

- The locations of edges/boundares will be preserved.

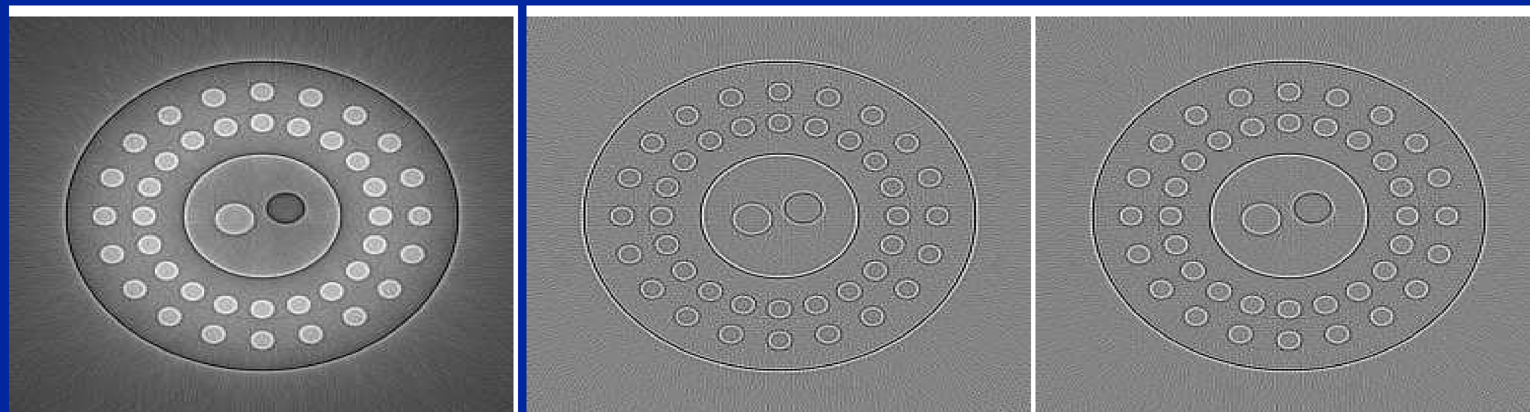
- It may not be possible to accurately estimate the magnitude of the reconstructed discontinuities.

Numerical Results

- We simulated phase-contrast projection data using a Fresnel wave propagation model.
- Untruncated and truncated projections sets were generated.

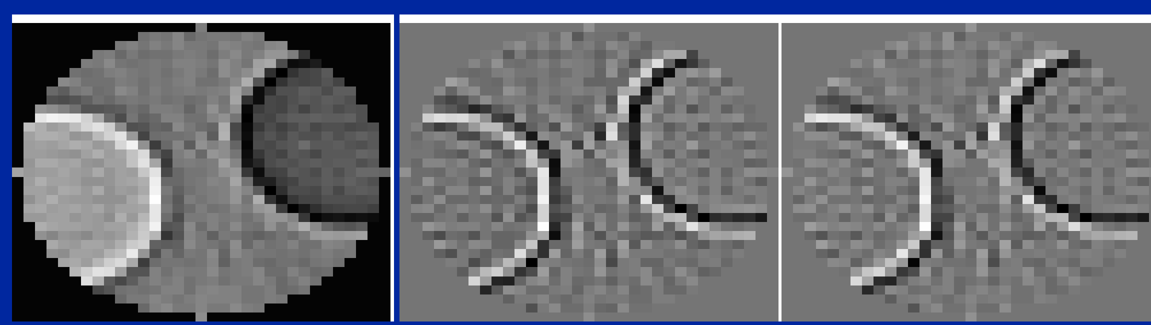


Reconstructions from untruncated projections:



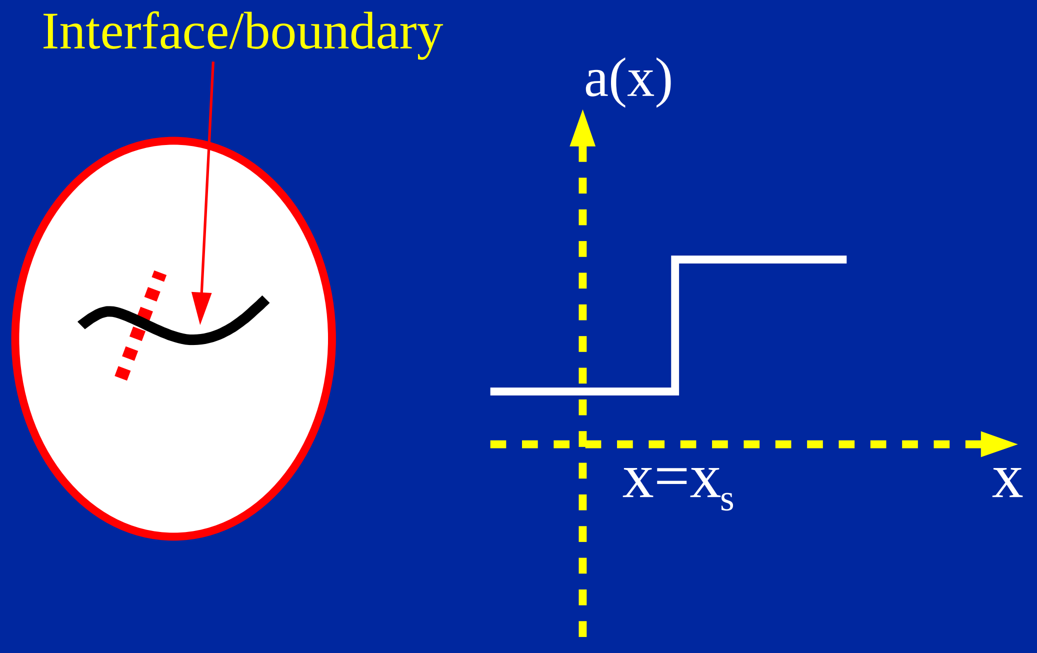
BP L-FBP FBP

Reconstructions from truncated projections:



Reconstructing the Magnitude of Discontinuities

- Knowledge regarding the magnitude of a discontinuity may be useful in many imaging applications.



$$D(x_s) = \lim_{t\to 0^+}\left[a(x_s+t) - a(x_s-t)\right]$$

Reconstructing the Magnitude of Discontinuities (2D Case)

- In the applied mathematics literature, several algorithms have been developed for estimating $D(x, y)$ from $\Lambda\mu(x, y)$

- As described previously, we can easily reconstruct $\Lambda a(\vec{r})$ from truncated phase-contrast projections.

- Therefore, we can reconstruct the magnitude of the 'jumps' in $a(\vec{r})$, in addition their location as indicated in the $\Lambda a(\vec{r})$ image.

- This information provides a useful characterization of the refractive index distribution.

Reconstructing the Magnitude of Discontinuities (Katsevich and Ramm, Local Tomo. and the Radon T.F.)

Numerical Algorithm:

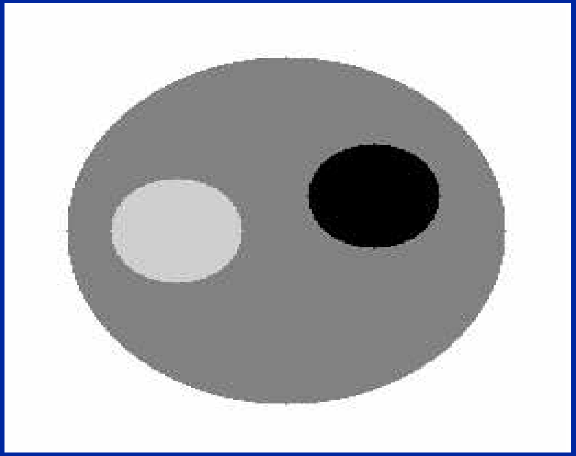
- Reconstruct $\Lambda a'(x, y) \equiv \Lambda a(x, y) ** w_\epsilon(x, y)$
- Compute normal vector to each boundary in $\Lambda a'(x, y)$
- Compute $D(x, y)$ by inverting the relationship

$$\Lambda a'(\vec{r_0}+t\epsilon n_0) = \frac{D(\vec{r_0})\psi(t)}{\pi\epsilon}(1+O(\epsilon))+\psi_\epsilon(\vec{r_0}, t)+O(\epsilon \ln \epsilon), \quad \epsilon \to 0$$

ψ, ψ_ϵ determined by w_ϵ.

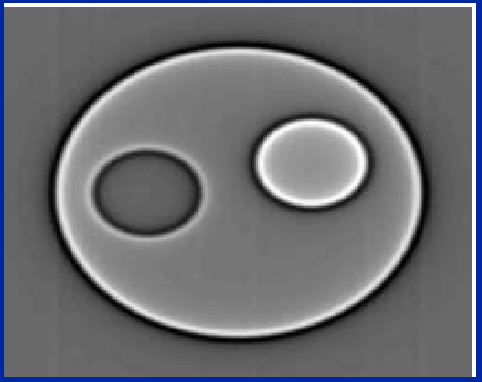
Simulation Studies

- Untruncated and truncated projections sets were generated corresponding to a numerical phantom.



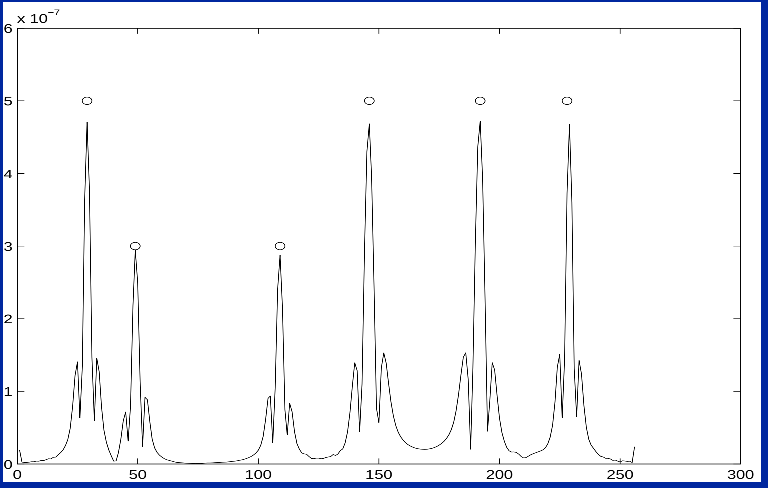
- We simulated 2D phase-contrast projection data using a Fresnel wave propagation model.

Reconstructions from untruncated projections:



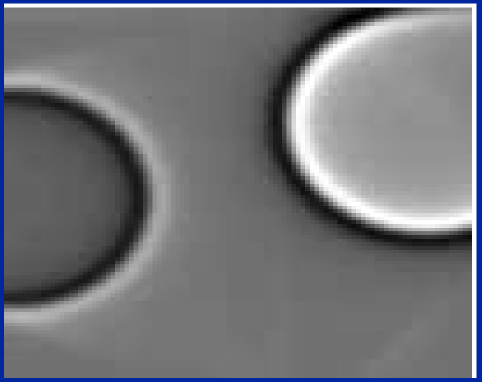
$\Lambda a'$

$D(x, y)$

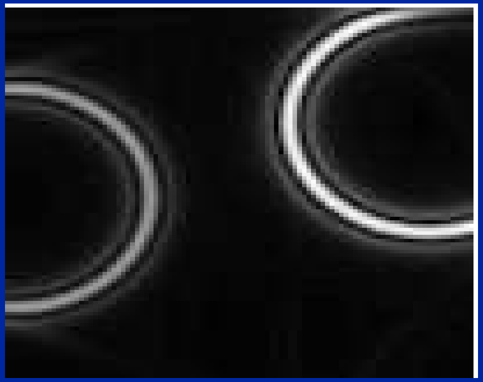


Profile through $D(x, 0)$

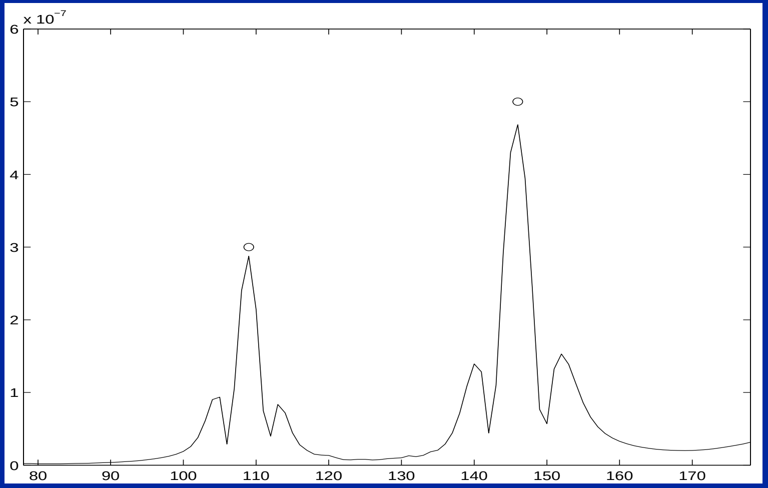
Reconstructions from truncated projections:



$\Lambda a'$

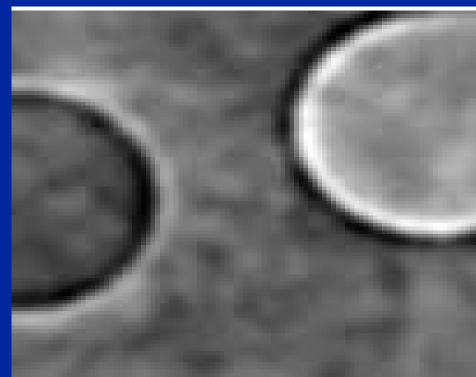


$D(x,y)$

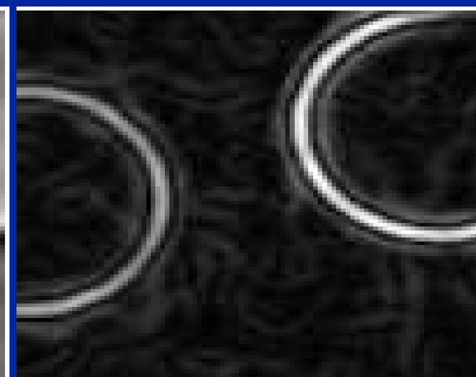


Profile through $D(x,0)$

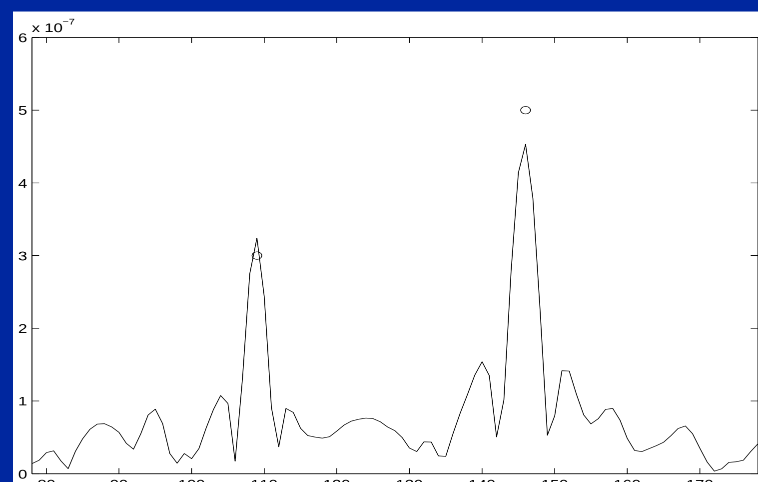
Reconstructions from noisy truncated projections:



$\Lambda a'$



$D(x, y)$



Profile through $D(x, 0)$

Summary

- We have investigated the local phase-contrast tomography problem.
- A simple backprojection of the data function represents an effective local tomography algorithm.
- For a wide class of objects, the FBP algorithm behaves like a local algorithm.
- For 2D problems, we can also reconstruct the magnitude of discontinuities.

  The 3D problem is a current research project.